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PROPAGATION OF WEAK DISCONTINUITIES FOR QUASI-LINEAR SYSTEMS

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The problem of propagation of weak discontinuities of solutions of quasi-linear hyperbolic systems is studied for the case when the characteristic surface satisfies an overdefined first order system of differential equations. A system of partial differential transport equations describing the propagation of a discontinuity along such characteristic surfaces is obtained. Systems of transport equations for a certain class of characteristic surfaces of the magnetic gasdynamics and crystal optics equations are given as an example.

A law of propagation of weak discontinuities over the region of constant motion, is obtained for a quasi-linear system with four independent variables which describes a number of processes in the continuum mechanics.

1. Consider an arbitrary quasi-linear hyperbolic system of equations given in the generalized sense [1]

$$L(\mathbf{U}) = \sum_{i=0}^{m} A^i \mathbf{U}_i + \mathbf{B} = 0 \tag{1.1}$$

$$\mathbf{U} = \{u_1, \ldots, u_n\},\ \mathbf{U}_j = \partial \mathbf{U} / \partial x_j,\ U_0 = \partial \mathbf{U} / \partial t \quad (j = 1, \ldots, m)$$

Here A^i denote matrices with elements $a^i_{\psi\varkappa}(x_j, t, \mathbf{U})$ and $\mathbf{B}$ is a vector with elements $b_\psi(x_j, t, \mathbf{U})$.

Let $\psi = \varphi(x_j) - t = 0$ denote the characteristic surface of the system (1.1). Then $A = A_1\varphi_1 + \ldots + A_m\varphi_m - A_0$ is the characteristic matrix of this system. When $t = 0$, the characteristic determinant

$$|A| = Q_1^{k_1} Q_2^{k_2} \cdots Q_\nu^{k_\nu} \qquad \left(1 \leqslant k_j \leqslant n,\ \sum_{j=1}^{\nu} k_j = n\right) \tag{1.2}$$

$$Q_\mu = \psi_0 + N_\mu(\varphi_j, x_j, t, \mathbf{U})$$

is a three-dimensional manifold. We shall assume that N_μ are continuously differentiable functions of x_j, t and φ_j in some region of the $(2m + 1)$-dimensional space G and that $N_\mu \not\equiv N_\omega$ in G for $1 \leqslant \omega$ and $\mu \leqslant \nu$ (here and henceforth we assume that the function of the solution has been inserted into the equations $Q_\mu = 0$).

Let the functions of the solution $\mathbf{U}$ be continuous on the passage across the characteristic surface $\psi = 0$, while the normal derivatives suffer a finite discontinuity (we denote it by $[\mathbf{U}_\varphi]$, $\mathbf{U}_\varphi = \partial \mathbf{U} / \partial \varphi$). Then [1]

$$[\mathbf{U}_\varphi] = \sum_{k=1} \sigma_k \mathbf{r}^k$$

Here σ_k are scalar functions and r^k are the right null vectors of the matrix A. By virtue of the hyperbolic character of (1.1), their number τ is equal to $(n - \lambda)$, λ

denoting the rank of the matrix A.

Let us select from the characteristic surfaces of the system (1.1) a bundle C of surfaces satisfying the equation $Q_\mu = 0$ (it can be assumed without loss of generality that $\mu = 1$), and segregate the surfaces belonging to C into classes, according to the rank of the matrix A associated with each surface. We find then that for the surfaces of class C_μ $(1 \leqslant \mu \leqslant \nu)$ we have $Q_1 = 0, \ldots, Q_\mu = 0$, and that by virtue of the hyperbolic character of (1.1), $\lambda = (n - k_1 - k_2 - \ldots - k_\mu)$ on such surfaces.

If the class C_s is nonempty (for some fixed $\mu = s$), two possibilities may arise:

a) the conditions of integrability hold for the system of equations

$$Q_1 = 0, \ \ldots, \ Q_s = 0 \tag{1.3}$$

i.e. the Poisson brackets constructed from the functions of this system are equal to zero (the system (1.3) is complete);

b) (1.3) is incomplete, but can be completed by incorporating the nonzero Poisson brackets in such a way that for the characteristic surfaces satisfying the resulting complete system we have

$$Q_1 = 0, \ldots, Q_s = 0, \ P_{s+1} = 0, \ldots, P_{s+k} = 0 \tag{1.4}$$
$$Q_\mu \neq 0, \quad \text{if} \quad s + 1 \leqslant \mu \leqslant \nu.$$

Depending on which of the two cases arises, let us solve the complete system (1.3) or (1.4) with respect to derivatives φ_a $(1 \leqslant a \leqslant l)$, assuming that the corresponding Jacobian is not zero in some region G_0 of the space G. The resulting system of equations

$$\varphi_a = f(\varphi_b, \ x_j, \ t) \qquad (1 \leqslant a \leqslant l, \ l + 1 \leqslant b \leqslant m, \ l \leqslant m) \tag{1.5}$$

will obviously be Jacobian, since it is equivalent to some complete system. When the case (b) applies, we have in (1.5) $l > s$, and $l \leqslant s$ for the case (a). Moreover, $l < s$ only when not all $Q_1, \ldots, Q_s$ are functionally independent.

We find that the pattern of propagation of the discontinuities remains similar for every class C_s.

For class C_1 it was studied in a number of papers [1 - 6]. We know that a discontinuity localized at some point of the surface belonging to C_1 propagates along the bicharacteristic ray of the initial system (1.1). The magnitude of the discontinuity is found at every instant either from the transport equation (when $k_1 = 1$), or from a set of transport equations (when $k_1 > 1$) [1 - 6].

The transport equations together with a set of equations of the bicharacteristics and equations describing the variation of φ_{ij} along the bicharacteristics $(\varphi_{ij} = \partial^2\varphi / \partial x_i \partial x_j)$ form a system of ordinary differential equations which can be solved in the general case by numerical methods, provided that the flow at one side of the characteristic surface for the quasi-linear system is also known (the flow parameters at this side will be denoted by a plus sign).

In Sect. 5 it will be shown that such a system of ordinary differential equations can be solved by analytic methods provided that

$$a_{\psi\varkappa} = a_{\psi\varkappa}(\mathrm{U}), \ b_\psi = b_\psi(\mathrm{U}), \ \mathrm{U}^+ = \mathrm{const}, \ \mathrm{U}_\varphi{}^+ = 0, \ m = 3 \tag{1.6}$$

The law of propagation of weak discontinuities coincides, under the conditions (1.6), if $b_\psi = \mathrm{const}$ and $k_1 = 1$, with the law of propagation of weak discontinuities for the gasdynamic equations [3, 4] with the accuracy of up to the constants.

If the surface S belongs to the class C_s $(1 < s \leqslant \nu)$, then a discontinuity originating at any point of such a surface propagates across an l -dimensional manifold where l denotes the number of equations in (1.5).

Sections 2 and 3 deal with the proof of this assertion. The law of propagation of the discontinuities for the class C_1 is obtained as a particular case of the investigation of an arbitrary class C_s when $l = 1$.

2. We shall show that the surfaces S satisfying the system (1.5) can be specified in a parametric form using l parameters. In other words, each parameter s_a $(1 \leqslant a \leqslant l)$ defines on the surface

$$x_i = g_i(s_1, \ldots, s_l), \qquad \varphi_i = \Phi_i(s_1, \ldots, s_l), \qquad \varphi_{ij} = \Phi_{ij}(s_1, \ldots, s_l) \qquad (2.1)$$

a curve whose direction at each point is given by

$$\mathbf{Y}_a = \{y_{a1}, \ldots, y_{am}\},\ y_{aj} = \delta_{aj},\ y_{ab} = -\partial f_a / \partial \varphi_b \ (1 \leqslant a, j \leqslant l;\ l+1 \leqslant b \leqslant m) \quad (2.2)$$

where δ_{aj} is the Kronecker delta.

Let us write for each equation of (1.5) a corresponding system of ordinary differential equations

$$\frac{dx_i}{ds_a} = y_{ai}, \qquad \frac{d\varphi_i}{ds_a} = \frac{\partial f_a}{\partial x_i} \qquad (i = 1, \ldots, m) \qquad (2.3)$$

This gives l systems of the form (2.3) and by virtue of the completeness of (1.5) the functions $F_a = \varphi_a - f_a(\varphi_b,\ x_i,\ t) = \text{const}$ represent the first integrals $(1 \leqslant a \leqslant l)$ for each of these sets. Separating from the solutions of (2.3) those for which the conditions $F_a = 0$ hold, we find that l curves can be drawn through any point of the surface S such that a planar element $\{x_i,\ \varphi_i\}$ at each point of these curves is an integral element of the system (1.5), and satisfies (2.3).

Let us consider a system of equations in $g_i\ (s_1, \ldots s_l)$ for a fixed value of i

$$\partial g_i / \partial s_a = y_{ai} \qquad (1 \leqslant a \leqslant l) \qquad (2.4)$$

We shall assume that the functions y_{ai} are continuously differentiable in some subregion S_0 of an m-dimensional space X with points $\{x_i\}$. Then the necessary and sufficient condition of solvability of (2.4) is valid in S_0. This is obvious for $1 \leqslant i \leqslant l$. For $i > l$ it follows from the fact that the Poisson brackets $(F_a F_j) \equiv 0$ when $1 \leqslant a$ and $j \leqslant l$. The completeness of (1.5) implies also the solvability of the system of equations

$$\partial \Phi_i / \partial s_a = \partial f_a / \partial x_i$$

for the function Φ_i, for every fixed value of i.

Assume that

$$x_i = g_i(s_1, \ldots, s_l), \qquad \varphi_i = \Phi_i(s_1, \ldots, s_l) \qquad (2.5)$$

Then the functions (2.5) satisfy (2.3). Consequently (2.5) hold on the surface S. We can prove in a similar manner that $\varphi_{ij} = \Phi_{ij}\ (s_1,\ \ldots,\ s_l)$ on S.

Note. We know [7] that under certain conditions imposed on the functions of (1.5), a solution of this system exists and is unique in the region $|x_a - \xi_a| < \varepsilon$ with $1 \leqslant a \leqslant l$ and for any x_b $(l+1 \leqslant b \leqslant m)$. This solution satisfies the initial value $\varphi(\xi_1, \ldots, \xi_l, x_{l+1}, \ldots, x_m) = \omega(x_{l+1}, \ldots, x_m)$. The set of systems (2.3) can be used to obtain the solutions for the systems of the form (1.5) in the manner similar to that in which the system of bicharacteristic equations is used for the case of a single first order equation. The solution of a system of partial differential equations is reduced to solving l

systems of ordinary differential equations.

3. Applying the procedures analogous to those in [6] (Sect. 1) to the case of the surface S belonging to the class C_s, we obtain the following system defining the scalar functions σ_k $(1 \leqslant k \leqslant \tau)$:

$$\sum_{k=1}^{\tau}\sum_{i=1}^{m} \mathbf{e}^j A^i \mathbf{r}^k \frac{\partial \sigma_k}{\partial x_i} + \Pi_j = 0 \qquad (j = 1, \ldots, \tau) \tag{3.1}$$

$$\Pi_j = \sum_{k=1}^{\tau} \sigma_k \mathbf{e}^j \Big[\sum_{i=1}^{m} A^i \mathbf{r}_i{}^k + (\nabla_{\mathbf{u}} L^{\varphi} \mathbf{r}^k) + A_{\varphi}\mathbf{r}^k + \sum_{\nu=1}^{\tau} \sigma_\nu (\nabla_{\mathbf{u}} A \mathbf{r}^\nu)\, \mathbf{r}^k\Big] \tag{3.2}$$

$$L^{\varphi} = A\mathbf{U}_{\varphi} + \sum_{i=1}^{m} A^i \mathbf{U}_i + \mathbf{B}, \qquad r^{ik} = \partial \mathbf{r}^k / \partial x_i$$

where $\mathbf{e}^j$ are the left null vectors of the matrix A.

Since on S the rank of A is equal to $\lambda = n - \tau$, a certain λ-th order minor of the matrix A is nonzero. We can assume without loss of generality that

$$M_{11} = M_{11}\begin{pmatrix} 1 \ldots \lambda \\ 1 \ldots \lambda \end{pmatrix}$$

It can be shown by direct computation that in (3.1)

$$\mathbf{e}^j A^i \mathbf{r}^k = \frac{\partial M_{jk}}{\partial \varphi_i} M_{11}, \qquad M_{jk} = M_{jk}\begin{pmatrix} j1 \ldots \lambda \\ k1 \ldots \lambda \end{pmatrix} \quad (j, k = \lambda + 1, \ldots, n) \tag{3.3}$$

Since $M_{jk}(f_a, \varphi_b, x_i, t) \equiv 0$, we have

$$\frac{\partial M_{jk}}{\partial \varphi_b} + \sum_{a=1}^{l} \frac{\partial M_{jk}}{\partial \varphi_a} \frac{\partial f_a}{\partial \varphi_b} = 0 \tag{3.4}$$

Using the relations (3.3), (3.4) and (2.1), we can write (3.1) in the form

$$\sum_{a=1}^{l}\sum_{k=1}^{\tau} \frac{\partial M_{jk}}{\partial \varphi_a} \frac{\partial \sigma_k}{\partial s_a} + \Pi_j (s_a, \sigma_k, \mathbf{U}^+, \mathbf{U}_i{}^+, \mathbf{U}_{\varphi}{}^+) = 0 \tag{3.5}$$

Functions Π_j $(s_a. \sigma_k, \mathbf{U}^+, \mathbf{U}_i{}^+, \mathbf{U}_\varphi{}^+)$ are obtained from (3.2) by substitution of the expressions (2.1).

The system (3.5) describes the transfer of a weak discontinuity along a characteristic surface of arbitrary class C_s and shows that the discontinuity arising at some point of S propagates through an l-dimensional manifold. The system, similarly to the system of transport equations for the class C_1, is not intrinsic, and makes possible the determination of the magnitude of the discontinuity only when the flow is known on one side of the surface.

4. In [6] the propagation of weak discontinuities is studied for the system of magnetogasdynamic equations in the case when the characteristic surface is adjacent to the region at rest and belongs to the class C_1. If the magnetic intensity vector $\mathbf{W} = \{h_1, h_2, h_3\}$ is parallel to the normal to the characteristic surface $\varphi(x_i) - t = 0$ and the speed of sound c is equal to the Alfvén velocity ($c^2 = \mu_l \mathbf{W}^2 / \rho$, $\mathbf{W}^2 = h_1{}^2 + h_2{}^2 + h_3{}^2$, where μ_l is the magnetic permeability and ρ is the density), the characteristic surface belongs to the class C_3, satisfies the system of equations $\varphi_i = h_i W^{-2} \rho^{1/2} \mu_i^{-1/2}$ $(i = 1, 2, 3)$ and becomes a plane when adjacent to the region at rest (h_i = const, ρ = const and $u_i = 0$ where u_i are the velocity vector components.)

Let us choose a coordinate system in such a manner that the equation of this plane has the form $x_1 - t = 0$. Then the propagation of weak discontinuities along such a characteristic surface is described by the following system of transport equations:

$$\frac{\partial \sigma_1}{\partial s_1} + \sum_{i=1}^{3} \left(\frac{\partial \sigma_i}{\partial s_i} + \sigma_i^2 \right) = 0 \qquad (x_i = s_i) \tag{4.1}$$

$$\frac{\partial \sigma_1}{\partial s_j} + 2 \frac{\partial \sigma_j}{\partial s_1} + \sigma_1 \sigma_j = 0 \qquad (j = 2, 3)$$

The system (4.1) is symmetrical hyperbolic and almost linear [1], and the plane $x_1 = s_1 = 0$ for this system is a surface of three-dimensional type. Therefore a solution to the Cauchy problem exists for (4.1) and is unique in some region R adjacent to the plane $x_1 = 0$, provided that the functions σ_i $(i = 1, 2, 3)$ are sufficiently smooth on this plane [1].

We shall show that under certain conditions imposed on the initial distribution of the weak discontinuities, the system (4.1) has a solution of the simple wave type near the initial manifold. Let us set $\sigma_2 = \psi_2(\sigma_1)$ and $\sigma_3 = \psi_3(\sigma_1)$. Then for σ_1 we obtain the following system of equations:

$$2 \frac{\partial \sigma_1}{\partial s_1} + \psi_2^{\cdot} \frac{\partial \sigma_1}{\partial s_2} + \psi_3^{\cdot} \frac{\partial \sigma_1}{\partial s_3} + \sigma_1^2 + \psi_2^2 + \psi_3^2 = 0$$

$$\frac{\partial \sigma_1}{\partial s_j} + 2\psi_j^{\cdot} \frac{\partial \sigma_1}{\partial s_1} + \sigma_1 \psi_j = 0 \qquad (j = 2, 3)$$

Assuming that $B = \psi_2^{\cdot 2} + \psi_3^{\cdot 2} \neq 1$, we can rewrite this system as follows:

$$\frac{\partial \sigma_1}{\partial s_i} = \frac{A_i}{2(B-1)} \qquad (i = 1, 2, 3) \tag{4.2}$$

$$A_1 = \sigma_1 \psi_3 \psi_3^{\cdot} + \sigma_1 \psi_2 \psi_2^{\cdot} - (\sigma_1^2 + \psi_2^2 + \psi_3^2)$$

$$A_j = 2 \left[\sigma_1 \psi_j \psi_{5-j}^{\cdot} + \psi_j^{\cdot} (\sigma_1^2 + \psi_2^2 + \psi_3^2) - \sigma_1 \psi_j - \sigma_1 \psi_{5-j} \psi_{5-j}^{\cdot} \psi_j^{\cdot} \right] \qquad (j = 2, 3)$$

where the dot indicates differentiation with respect to σ_1. The system (4.2) has a solution if $A_\nu^{\cdot} A_\mu - A_\nu A_\mu^{\cdot} = 0$ $(\nu, \mu = 1, 2, 3)$. These conditions hold if and only if ψ_2 and ψ_3 satisfy the following system of ordinary differential equations

$$A_1 = M_1 A_2, \qquad A_1 = M_2 A_3 \tag{4.3}$$

where M_1 and M_2 are constants. Since (4.3) has a solution dependent on arbitrary constants, the system (4.1) has a solution of the simple wave type under an arbitrary choice of constants.

If conditions (4.3) hold, the function σ_1 satisfies

$$M_1 \frac{\partial \sigma_1}{\partial s_2} - \frac{\partial \sigma_1}{\partial s_1} = 0, \qquad M_2 \frac{\partial \sigma_1}{\partial s_3} - \frac{\partial \sigma_1}{\partial s_1} = 0$$

This implies that the planes

$$t + x_2 M_1^{-1} + x_3 M_2^{-1} = \text{const} \tag{4.4}$$

are equipotential surfaces for σ_1 and that a solution of (4.1) of the simple wave type exists near the plane $x_1 = 0$, provided that the weak discontinuities are distributed on the initial manifold $x_1 = 0$ in such a manner that $\sigma_i = \text{const}$ along the straight lines $M_2 x_2 + M_1 x_3 = \text{const}$. A weak discontinuity appearing on the line $M_2 x_2 + M_1 x_3 = \text{const}$ appears at every point of the plane (4.4) and fills a two-dimensional manifold. In the general case, a weak discontinuity can obviously propagate from every point of the initial plane $x_1 = 0$ over a three-dimensional manifold.

We have an analogous case in crystal optics, where the normal to the characteristic surface plan is directed along the optical axis of the crystal. The characteristic surface satisfies the following system of equations [8]

$$\varphi_1 = \left[\frac{\mu_l \varepsilon_3 (\varepsilon_1 - \varepsilon_2)}{\rho^2 (\varepsilon_1 - \varepsilon_3)}\right]^{1/2}, \quad \varphi_2 = 0, \quad \varphi_3 = \left[\frac{\mu_l \varepsilon_1 (\varepsilon_2 - \varepsilon_3)}{\rho^2 (\varepsilon_1 - \varepsilon_3)}\right]^{1/2}$$
$$(\varepsilon_1 > \varepsilon_2 > \varepsilon_3)$$

where ε_1, ε_2 and ε_3 are the dielectric constants in the directions of the coordinate axes.

The system of transport equations has the form

$$2\varepsilon_3 \frac{\partial \sigma_1}{\partial s_1} - 2\varepsilon_2 \varepsilon_3 \frac{\partial \sigma_1}{\partial t} + (\varepsilon_2 - \varepsilon_3)\varphi_1 \frac{\partial \sigma_2}{\partial s_2} = 0$$
$$(\varepsilon_1 - \varepsilon_3)\frac{\partial \sigma_1}{\partial s_2} - 2\varepsilon_1 \varphi_1 \frac{\partial \sigma_2}{\partial s_1} - 2\varepsilon_1 \varepsilon_3 \frac{\partial \sigma_2}{\partial t} = 0$$

and can be easily reduced to a wave equation for each σ_i $(i = 1, 2)$

$$\frac{2\mu_l^2 \varepsilon_2 \varepsilon_3^2 (\varepsilon_1 - \varepsilon_3)}{c^4 (\varepsilon_2 - \varepsilon_3)(\varepsilon_1 - \varepsilon_2)^2} \frac{\partial^2 \sigma_1}{\partial \nu^2} + \frac{1}{2\varepsilon_1} \frac{\partial^2 \sigma_1}{\partial s_2^2} - \frac{2\mu_l \varepsilon_2 \varepsilon_3}{c^2 (\varepsilon_2 - \varepsilon_3)(\varepsilon_1 - \varepsilon_2)} \frac{\partial^2 \sigma_1}{\partial t^2} = 0$$
$$\frac{2\mu_l \varepsilon_1^3 \varepsilon_2 (\varepsilon_2 - \varepsilon_3)^2}{c^4 \varepsilon_3 (\varepsilon_1 - \varepsilon_3)(\varepsilon_1 - \varepsilon_2)^2} \frac{\partial^2 \sigma_2}{\partial z^2} + \frac{\varepsilon_2 - \varepsilon_3}{2\varepsilon_3} \frac{\partial^2 \sigma_2}{\partial s_2^2} - \frac{2\mu_l \varepsilon_1 \varepsilon_2}{c^2 (\varepsilon_1 - \varepsilon_2)} \frac{\partial^2 \sigma_2}{\partial t^2} = 0$$

where

$$t = \varphi_1 s_1 + \varphi_3 s_3, \qquad x_i = s_i \qquad (i = 1, 2, 3)$$
$$\nu = \frac{\varepsilon_3 (\varepsilon_2 + \varepsilon_3)}{\varphi_1 (\varepsilon_2 - \varepsilon_3)} t - \frac{2\varepsilon_2 \varepsilon_3 (\varepsilon_1 - \varepsilon_3)}{(\varepsilon_1 - \varepsilon_2)(\varepsilon_2 - \varepsilon_3)} s_1$$
$$z = \frac{\varepsilon_1 (\varepsilon_2 + \varepsilon_3)}{\varphi_1 (\varepsilon_1 - \varepsilon_3)} t - \frac{2\varepsilon_1 \varepsilon_2}{\varepsilon_1 - \varepsilon_2} s_1$$

This confirms the well known effect of conical refraction which occurs when the normal to the characteristic surface has the orientation indicated above.

5. We shall consider the system (1.1) with conditions (1.6). We assume that the characteristic surface $\varphi(x_i) - t = 0$ satisfies the equation $Q_1 = 0$ and that in (1.2) $k_1 = 1$. Under these conditions $[\mathbf{U}_\varphi] = \sigma \mathbf{r}$ (where $\mathbf{r}$ is the right null vector of the matrix A and σ is a scalar function) and the transport equation for (1.1) has the form [6]

$$\alpha \frac{d\sigma}{dt} + \sigma \left[\mathbf{e} \sum_{i=1}^{m} A^i \mathbf{r}_i + \mathbf{e} \sum_{\varkappa=1}^{n} \mathbf{B}_\varkappa r^\varkappa\right] + \sigma^2 [\mathbf{e}(\nabla_u A \mathbf{r})\mathbf{r}] = 0 \tag{5.1}$$

here

$$\alpha = A_{11} Q_2 \cdots Q_n = \alpha(\mathbf{U}, \varphi_i), \quad \mathbf{B}_\varkappa = \partial \mathbf{B} / \partial u_\varkappa, \quad \mathbf{r}_i = \partial \mathbf{r} / \partial x_i$$

$\mathbf{e}$ is the left null vector of A, $r^\varkappa$ are the components of the vector $\mathbf{r}$, and A_{11} is the algebraic complement of the element a_{11} of the matrix A (it is assumed that $A_{11} \neq 0$).

Let us write the system of bicharacteristic equations for (1.1)

$$dx_i/dt = \partial Q_1/\partial\varphi_i, \qquad d\varphi_i/dt = 0 \qquad (i = 1, \ldots, m) \tag{5.2}$$

The functions

$$\varphi_i = \text{const}, \ x_i = (\partial Q_1 / \partial\varphi_i)\, t + M_i$$

where $\partial Q_1 / \partial\varphi_i$ and M_i are constants, represent a solution of this system, i.e. the bicharacteristics are straight lines.

Since $\varphi_i = \text{const}$, we have in (5.1)

$$\alpha = \text{const}, \ \mathbf{e} \sum_{\varkappa=1}^{n} \mathbf{B}_\varkappa \mathbf{r}^\varkappa = T_1 = \text{const}, \ \mathbf{e}\,(\nabla_\mathbf{u} A\mathbf{r})\,\mathbf{r} = T_2 = \text{const}$$

$$r_i = \sum_{k=1}^{n} (\partial\mathbf{r} / \partial\varphi_k)\, \varphi_{ki}$$

As we know,

$$\frac{d\varphi_i}{dt} = \sum_{j=1}^{m} \frac{\varphi_{ij}\partial Q_1}{\partial\varphi_j} = 0$$

and this gives

$$\varphi_{im} = -\Big(\sum_{j=1}^{m-1} \frac{\varphi_{ij}\partial Q_1}{\partial\varphi_j}\Big) \Big/ \Big(\frac{\partial Q_1}{\partial\varphi_m}\Big) \tag{5.3}$$

Differentiating the equation $Q_1 = 0$ with respect to x_i and x_j $(i, j = 1, \ldots, m-1)$ and inserting the values given by (5.3), we obtain the following system defining $\varphi_{ij}(t)$

$$\frac{d\varphi_{ij}}{dt} = \sum_{k,\nu=1}^{m-1} \alpha_{k\nu}\varphi_{ik}\varphi_{j\nu} \tag{5.4}$$

$$\alpha_{k\nu} = -\frac{\partial^2 Q_1}{\partial\varphi_k\partial\varphi_\nu} + \frac{\partial^2 Q_1}{\partial\varphi_k\partial\varphi_m}\frac{\partial Q_1/\partial\varphi_\nu}{\partial Q_1/\partial\varphi_m} + \frac{\partial^2 Q_1}{\partial\varphi_\nu\partial\varphi_m}\frac{\partial Q_1/\partial\varphi_k}{\partial Q_1/\partial\varphi_m} -$$
$$- \frac{\partial^2 Q_1}{\partial\varphi_m^2}\frac{\partial Q_1/\partial\varphi_k \partial Q_1/\partial\varphi_\nu}{(\partial Q_1/\partial\varphi_m)^2} = \text{const}$$

When $m = 3$, the system (5.4) has the first integrals

$$\frac{\alpha_{12}\varphi_{22} + \alpha_{11}\varphi_{12}}{\alpha_{12}\varphi_{11} + \alpha_{22}\varphi_{12}} = N_1, \qquad \frac{\alpha_{12}\varphi_{11} + \alpha_{22}\varphi_{12}}{\alpha_{12}(\varphi_{12}^2 - \varphi_{11}\varphi_{22})} = N_2 \tag{5.5}$$

which can be used to reduce it to a single equation

$$\frac{d\varphi_{11}}{dt} = \frac{\alpha_{22}X + \beta X^{1/2}}{2N_2^2\alpha_{12}^2}$$

$$X = [\alpha_{22} - N_2\varphi_{11}(\alpha_{11} - N_1\alpha_{22})]^2 + 4N_2\alpha_{12}\varphi_{11}(N_1N_2\varphi_{11} + 1) \tag{5.6}$$
$$\beta = 2N_2\alpha_{12}^2\varphi_{11} + \alpha_{22}^2 - N_2\alpha_{22}\varphi_{11}(\alpha_{11} - N_1\alpha_{22})$$

Solution of (5.6) has the form

$$\varphi_{11} = \frac{a_1(\alpha_{ij}) + a_2(\alpha_{ij})(t + T_3)}{a_3(\alpha_{ij}) + a_4(\alpha_{ij})(t + T_3)^2} \tag{5.7}$$

where T_3 is an arbitrary constant, and the values of the remaining $\varphi_{ij}(t)$ are defined after inserting (5.7) into (5.5) and (5.3).

It can easily be verified by direct computation that in (5.1)

$$\mathbf{e}\sum_{i=1}^{m} A^{i}\mathbf{r}_i = -\frac{1}{2}\alpha \sum_{i,k=1}^{m} \alpha_{ik}\varphi_{ik} \tag{5.8}$$

Taking (5.8) into account, let us insert into (5.1) the values of $\varphi_{ik}(t)$. For $m = 2$ we have $\varphi_{ik} = a_{ik} / (b_{ik} - t)$, where a_{ik} and b_{ik} are constants. For $m = 3$ we utilize (5.7), (5.5) and (5.3), obtaining the following transport equation

$$d\sigma/dt + \sigma[\Lambda_1 + {}^1/_2\Lambda(t)] + \Lambda_2\sigma^2 = 0$$

$$\Lambda_1 = \frac{T_1}{\alpha}, \quad \Lambda_2 = \frac{T_2}{\alpha}, \quad \Lambda(t) = \begin{cases} (t+T_0)^{-1} + (t+T^0)^{-1}, & m=3 \\ (t+T_4)^{-1}, & m=2 \end{cases} \tag{5.9}$$

$$\sigma^{-1} = e^{\Lambda_1 t} f(t) \left[M + \Lambda_2 \int e^{-\Lambda_1 t} f^{-1} dt\right]$$

$$f(t) = \begin{cases} (t+T_0)^{1/2}(t+T^0)^{1/2}, & m=3 \\ (t+T_4)^{1/2}, & m=2 \end{cases} \tag{5.10}$$

where M is an arbitrary constant.

If in (1.1) $b_\psi = \text{const}$, then in (5.10) $\Lambda_1 = 0$. Thus the law of variation of $\sigma(t)$ with $b_\psi = \text{const}$ coincides, with the accuracy of up to the constant terms, with the gasdynamic laws of propagation of weak discontinuities [3, 4].

Since the form (1.1) with $b_\psi = b_\psi(\mathbf{U})$ and $a_{\psi\varkappa} = a_{\psi\varkappa}(\mathbf{U})$ is shared by numerous systems describing processes of continuum mechanics (plasticity, gasdynamics, magnetogasdynamics and others), the laws of propagation of discontinuities (5.10) are universal.

N o t e . In the transport equation for the system of gasdynamic equations the value of the coefficient of σ coincides with the mean curvature H of the surface $\varphi(x_i) = t$ [4]. This is however not true for any arbitrary system (1.1). For example, in the case of magnetogasdynamics equations we have

$$\sum_{i,k=1}^{2} \alpha_{ik}\varphi_{ik} = 4|\varphi|^{-3}\left\{H - 2c^2\varphi_3\varphi_{11}|\varphi|^{-3}\frac{\mu_i}{\rho}\left(3c^2\varphi_1^2\frac{\mu_i}{\rho} + |\varphi|^{-2}\right)\right\}$$

$$|\varphi| = \left(\sum_{i=1}^{3}\varphi_i^2\right)^{1/2}$$

and the coefficient accompanying σ (see (5.9)) is not equal to H.

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ON CERTAIN CLASSES OF QUASI-STATIONARY FLOWS OF A PERFECT INCOMPRESSIBLE FLUID

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We consider a nonstationary flow with stationary streamlines (i.e. a quasi-stationary flow) of a perfect incompressible fluid in a conservative external force field.

A specific property is obtained for the field of velocity directions of an irrotational quasi-stationary flow, a relationship determined between the moduli of the velocities of the quasi-stationary and the stationary flow with the same streamlines, and a possibility of existence of rotational and irrotational quasi-stationary flows with common streamlines is studied.

In [1 – 3] the necessary conditions are obtained for the field of unit vectors in order that it may serve as a field of velocity directions of stationary flow of an incompressible fluid. An analogous problem for a quasi-stationary flow is solved in [4] only for the case when the field of velocity directions is rectilinear.

1. Let us denote the unit velocity direction vector by $\mathbf{e}$ and the streamline curvature vector by $\mathbf{k}$. The field of vectors $\mathbf{l} = \mathbf{k} - \mathbf{e} \operatorname{div} \mathbf{e}$ is called the field of adjoint vectors of the field of $\mathbf{e}$. A vector field is called holonomic [5], if there exists a family of surfaces orthogonal to the field. The quantity $H = \operatorname{div} \mathbf{e}$ is the mean curvature of the field of $\mathbf{e}$ [6]. A field of mean zero curvature is called the minimal field [1].

We shall now find the necessary and sufficient geometrical conditions for the field of unit vectors $\mathbf{e}$ in order that it may serve as a field of velocity directions of an irrotational quasi-stationary flow.

Theorem 1. The field of unit vectors $\mathbf{e}$ may serve as a field of velocity directions of an irrotational quasi-stationary flow if and only if

1) the field of $\mathbf{e}$ is holonomic ($\mathbf{e} \cdot \operatorname{rot} \mathbf{e} = 0$);

2) the field of its adjoint vectors is potential ($\operatorname{rot} \mathbf{l} = 0$).